Original Article

Providing a Numerical Method for Solving Groundwater Equations

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ABSTRACT

Network methods are one of the most widely used groundwater modeling tools that have become widespread and popular in the last two decades. On the other hand, advances in the computing power of computers and their ease of access have led to the rapid development of numerical methods for solving groundwater problems. In this study, with a new approach to network methods, these methods are introduced as a numerical model to simulate the movement of groundwater. The first step in estimating groundwater behavior is to obtain a mathematical model. According to Darcy's law and the establishment of mass conservation, it can be shown that the equation governing groundwater in stable conditions is the Laplace equation. Therefore, by obtaining the physical properties of the desired aquifer using the experiment and understanding the boundary conditions governing that aquifer, a complete mathematical model governing the desired problem can be obtained. Unfortunately, since the physical properties of the problems in nature are not homogeneous and the boundaries of the problems under study are geometrically irregular, it is impossible to solve these problems analytically. To do so, researchers have been using numerical and laboratory methods for years. All numerical methods for solving the Laplace equation first decompose the scope of the problem in the sense that they either divide it into multiple nodes or into multiple elements and then use mathematical methods. In this article, a new approach is to find algebraic relations between those different nodes or elements. In other words, after parsing the scope of the problem, they turn the Laplace equation into a system of linear equations.

Introduction

Flow simulation in porous media has attracted the attention of many researchers over the past three decades. Various applications of this simulation can be found in fields such as water engineering, environmental engineering, petroleum engineering, and hydrology [1-3].

Groundwater pumped from underground structures is the main source of many water

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source systems. The amount of water discharged from a spring is considered as the output of the groundwater system. It can be greatly affected by the amount of pumping carried out from the same area. For storage, water can be injected into wells drilled for this purpose, and groundwater levels can be raised using the same technique. These are some of the issues that can affect the management of groundwater [4-6].

In fact, in groundwater management system, quality and quantity issues cannot be considered as a separate issue. In many parts of the world, due to excessive groundwater abstraction, the quality of this water is constantly deteriorating, which has caused the attention of consumers and producers to this issue. Groundwater degradation can be due to increased water salinity or ionic concentrations such as nitrate.

In recent years, in addition to the general issues affecting water quality, public attention has been paid to the issue of groundwater pollution by toxic industrial wastewater, leachate from landfills, petroleum products and other toxic liquids, fertilizers, herbicides and insecticides used in agriculture and radioactive materials buried deep underground have been targeted. Although many of these problems occur on the earth's surface, these pollutants enter groundwater after penetrating the earth [7-9].

After joining groundwater, these pollutants are transported by groundwater movement and reach rivers, lakes and wells. On the other hand, the limited groundwater has made the groundwater increasingly important as a source of human drinking water. Any planning for control and clearing operations requires the estimation and estimation of the values under study. Consequently, any operation to read the values under study requires knowledge of how groundwater behaves.

Therefore, proper management is achieved when the response of the system to the activities can be considered. One of the initial steps required to estimate the behavior of groundwater is to find a mathematical model, the application of which, in turn, requires the collection of information. The more accurate the information collected, the more reliable the model answers can be, although data collection is always fraught with error and uncertainty [10].

Using experimental systems, the cause of errors caused by human error or human ignorance can be greatly reduced and eliminated. Therefore, many studies have been done on issues such as maintenance of a wellhead, design of drinking water supply systems, estimation of movement and transfer of pollutants in the aquifer and mathematical models for these problems have been presented [11].

The use of grid models to simulate flow within a porous medium has grown dramatically over the past decade, and many researchers have studied these grids using different tubes and cavities.

However, none of these studies has been used to investigate the PNM method as a numerical tool for solving groundwater problems [12]. The innovation of this study is to introduce a new approach to PNM methods as a numerical method and replacing them with conventional numerical methods such as finite difference and finite elements [13].

In this article, a new approach is to find algebraic relations between those different nodes or elements. In other words, after parsing the scope of the problem, they turn the Laplace equation into a system of linear equations.

Materials and Methods

Numerical and Laboratory Examples and Discussion of the Results Obtained

Conventional numerical methods for solving groundwater equations are two methods of finite difference and finite elements, each of which has advantages and disadvantages over each other.

For example, the advantage of the finite element method over the finite difference is the ability to model aquifers with boundaries that are not geometrically regular. The main advantage of the finite difference method is its simplicity in application and obtaining accuracy that is comparable to the finite element method [14].

Laboratory Examples

This section deals with the results obtained from the laboratory device. Since this laboratory device consists of only tubes that are connected horizontally and vertically, in this section, the results obtained from the experiment are compared with the SPNM method and the numerical methods FD and FE [15].

The grids that make up this device are all square with dimensions of 2.1 cm. This device is connected to the two power supplies from the left and right to adjust the head.

Experiment (1): Flow around a rectangular barrier

This experiment examines the movement of water around an impermeable rectangle located in an enclosed aquifer. The left and right boundaries of the aquifer have a fixed head and are equal to 160 cm and 110 cm, respectively, and the two upper and lower boundaries of the aquifer are impermeable. The shape of this aquifer is shown below.

Using the shown division, different nodes can be obtained using three methods: FE, FD, and SPNM. Because there is no analytical answer to this problem, this problem is solved numerically by FD and FE methods and the problem divisions are so small that the obtained answers do not change significantly as the divisions become smaller.

These results are then assumed to be the exact answer to the problem. The results obtained using the above three methods are reported in Table 1.

To perform this test, all pipes that are in the impermeable area are removed. The schematic of this experiment is shown in Figure 2.

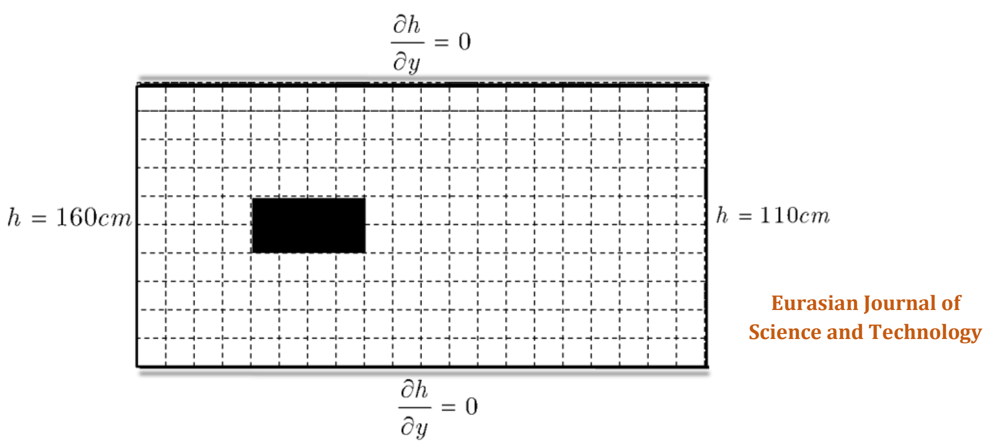

Figure 1 Experimental model of water movement around an impermeable rectangle

Table 1 Mean percentage of relative error obtained by the division using different numerical methods for Experiment (1)

Mesh size	TAE		
	FD	FE	SPNM
2×1 cm	1.66	0.42	1.50

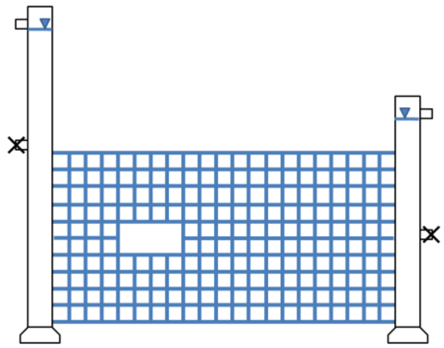

Figure 2 Schematic view of water movement test around an impermeable rectangle

To measure the pressure in the experimental model, piezometers were installed in different parts of the experimental model. Because these piezometers were installed in the middle of horizontal or vertical tubes, the read heads would not be aligned with the computing heads [16].

Therefore, to obtain the computational heads of the points where the piezometer is installed, the average computational head of two adjacent nodes should be used. Laboratory error was calculated in this way and presented in Table 2. Since the number of nodes to obtain the laboratory error is not the same as the number of nodes to obtain the error by numerical methods, merging Tables 1 and 2 was avoided.

Table 2 Mean percentage of relative error obtained by the division using Experiment (1)

Mesh sizes	TAE of Experiment
2×1 cm	10

As can be seen, the error obtained from the SPNM network method and the test results are of acceptable accuracy. In Figure 3, to compare the obtained results, lines of both potential and velocity vectors of water are drawn around the impermeable rectangle. The velocity vectors well represent the passage of water around the barrier.

This is very similar in the forms obtained from different numerical methods. It should be noted that since in such experiments the number of points at which the head of the laboratory device was read (50 knots) was less than the number of knots at which the head was calculated (200 knots), both potential lines and velocity vectors were used only for methods. Computations were drawn.

Experiment (2): Flow with composite boundary conditions

This experiment examines the movement of water in a confined aquifer with combined boundary conditions. The two upper and lower boundaries are impermeable in this case, as in the previous example, but the left and right boundaries each have fixed-head sections and an impermeable boundary. The shape of this experiment and a schematic view of the laboratory device are demonstrated in Figures 4 and 5.

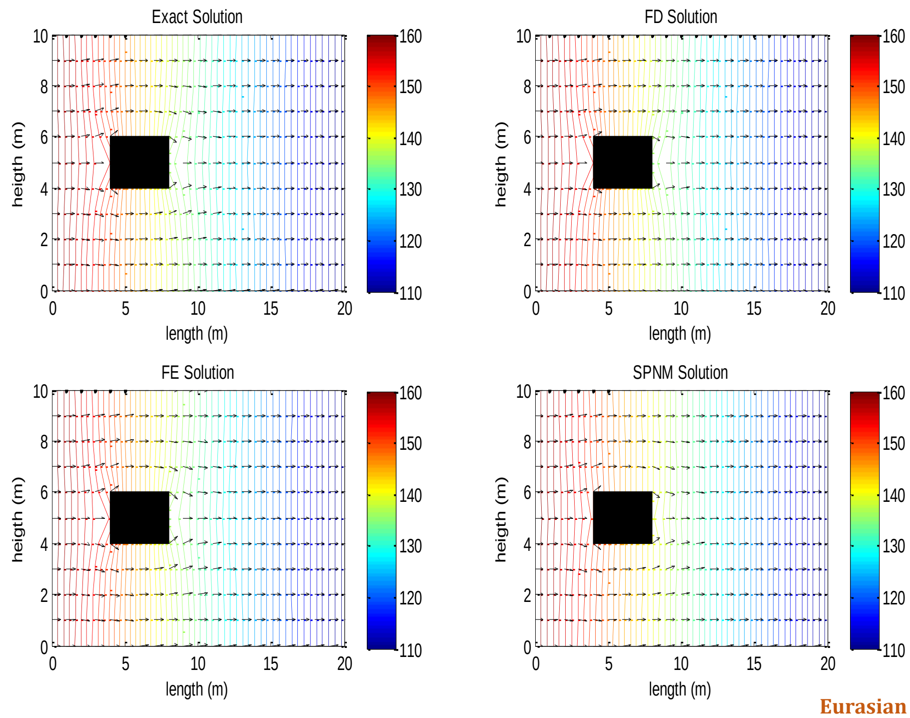

Figure 3 Co-potential lines and velocity vectors of water around an impermeable rectangle

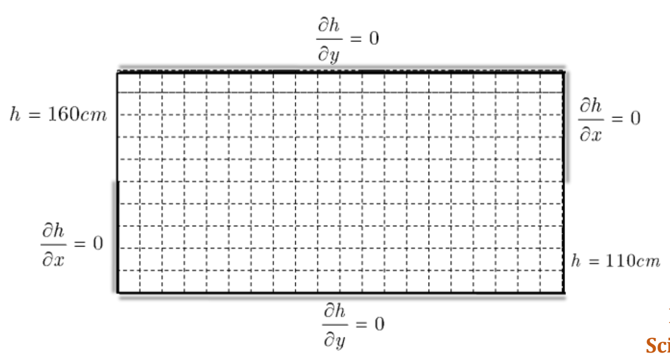

Figure 4 Experimental model of water movement in enclosed aquifers with combined boundary conditions

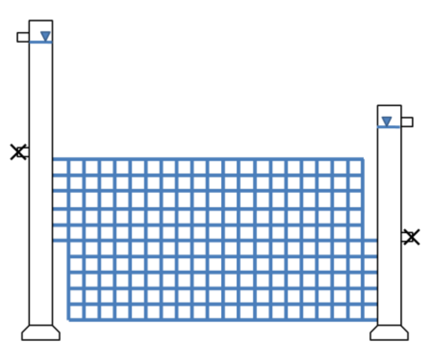

Figure 5 Schematic view of the experiment with combined boundary conditions

The analytical answer to this problem using the method described in the previous example was compared with the answers obtained using the three methods FD, FE, and SPNM, and also the resulting errors are reported in Table 3. Furthermore, laboratory errors are listed in Table 4.

Table 3 Mean percentage of relative error obtained by division using different numerical methods for Experiment (2)

Mesh size	TAE		
	FD	FE	SPNM
2×1 cm	0.75	0.80	0.60

Table 4 Mean percentage of relative error obtained by the division using Experiment (2)

Mesh sizes	TAE of Experiment
2×1 cm	8

In Figure 6, the passage of water in a curved path that starts from the upper left of the laboratory device and reaches the lower right of the instrument is evident using different numerical methods.

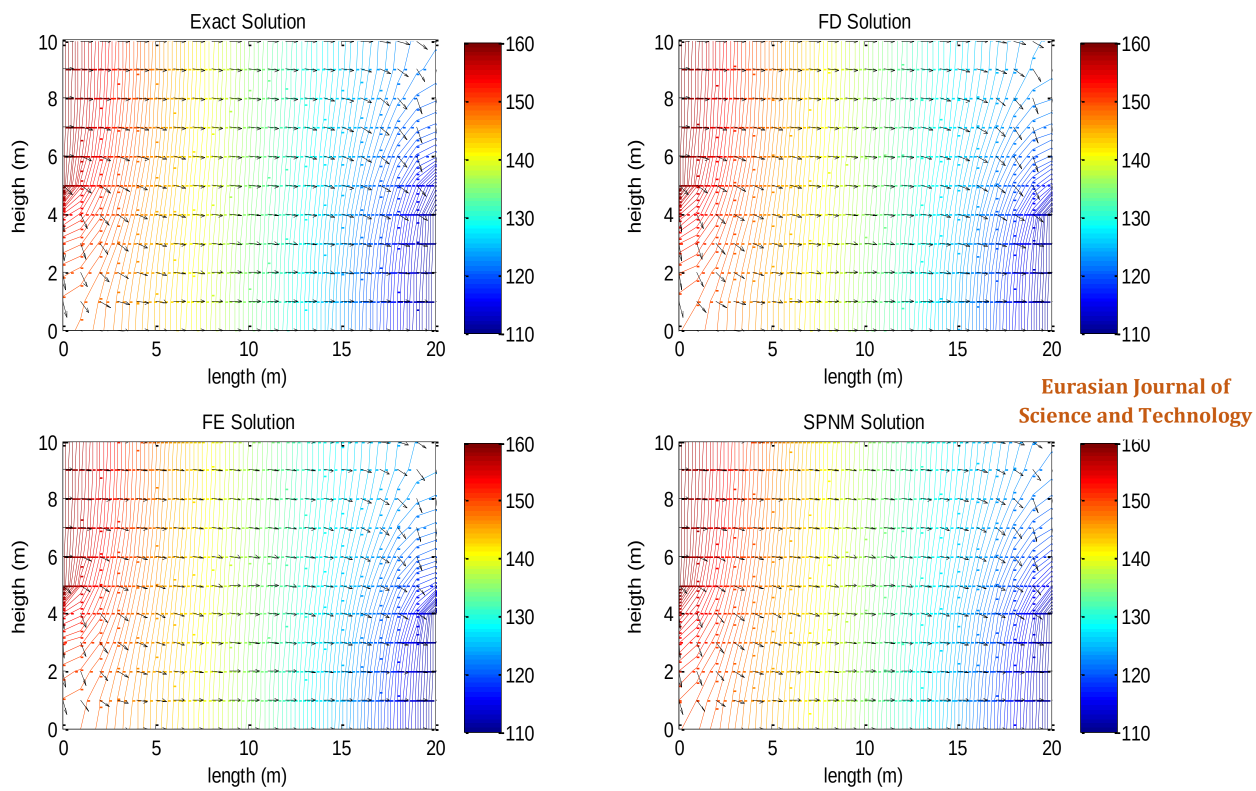

Figure 6 Co-potential lines and velocity vectors in Experiment (2)

Experiment (3): Flow from under the sealing curtain

This experiment simulates the movement of water under a dam despite the sealing curtain. The height of water upstream is equal to 160 cm and downstream is equal to 110 cm [17].

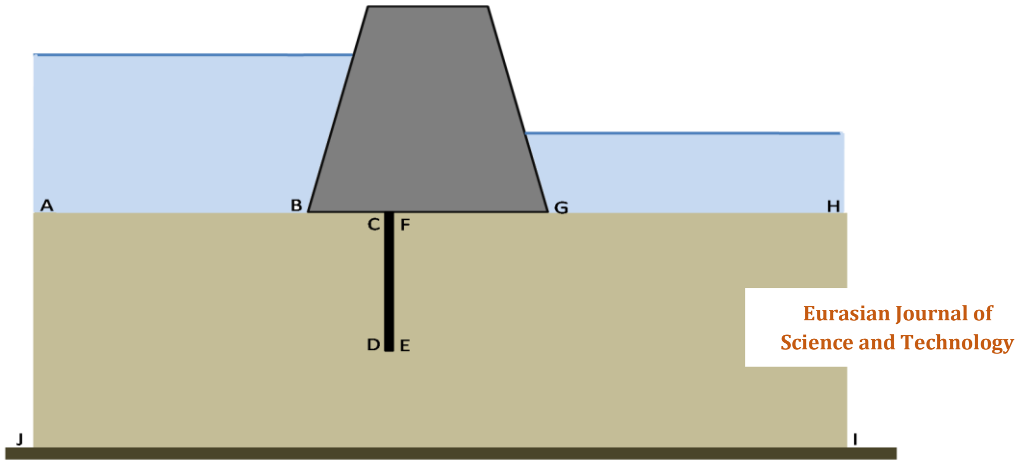

Figure 7 Model of water movement under a dam despite the sealing curtain

Since the boundaries AB and GH are longitudinally infinite, they cannot be considered in the same way in the numerical or experimental model. The method proposed for these problems is that by replacing these two boundaries of limited length in the problem, it is solved [18].

Then the length of these two boundaries is added and the problem is solved again. This continues until there is no change in the answers obtained near the sealing curtain. Because the lengths of the AB and GH boundaries are much larger than other sizes using this method, which makes for a much larger laboratory, a porous medium is needed to test the motion of water. From under the curtain, the seal was considered as follows [19].

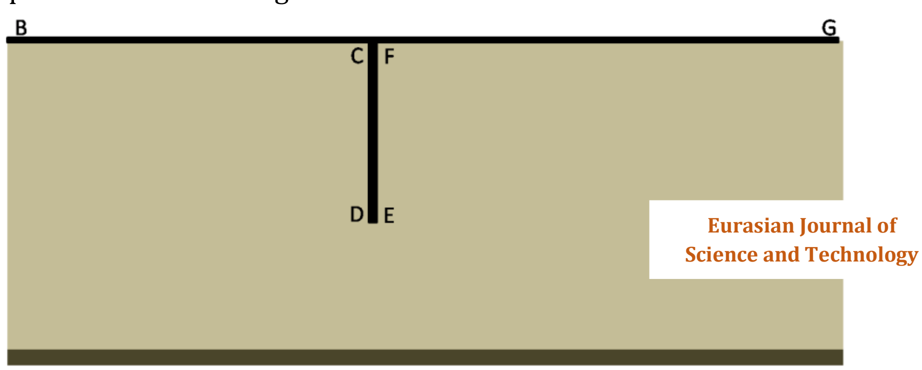

Figure 8 Amplitude intended for testing 3

In this example, the left and right boundaries were assumed to have a fixed head of 160 and 110 cm, respectively, and the other boundaries were all considered to be impenetrable. The height of this aquifer was considered equal to 1 meter and its length was equal to 2 meters.

The height of the sealing curtain is 0.6 meters and its thickness is 0.1 meters. The BC border length was assumed to be 0.6 m. To perform the experiment, the left and right connecting channels of the sealing curtain were removed to model the sealing curtain. The method of performing the experiment and its computational range is illustrated in Figure 9.

Furthermore, Figure 10 shows the potential lines and velocity vectors obtained through different methods. The way water moves around the sealing curtain in these figures shows quite a trend.

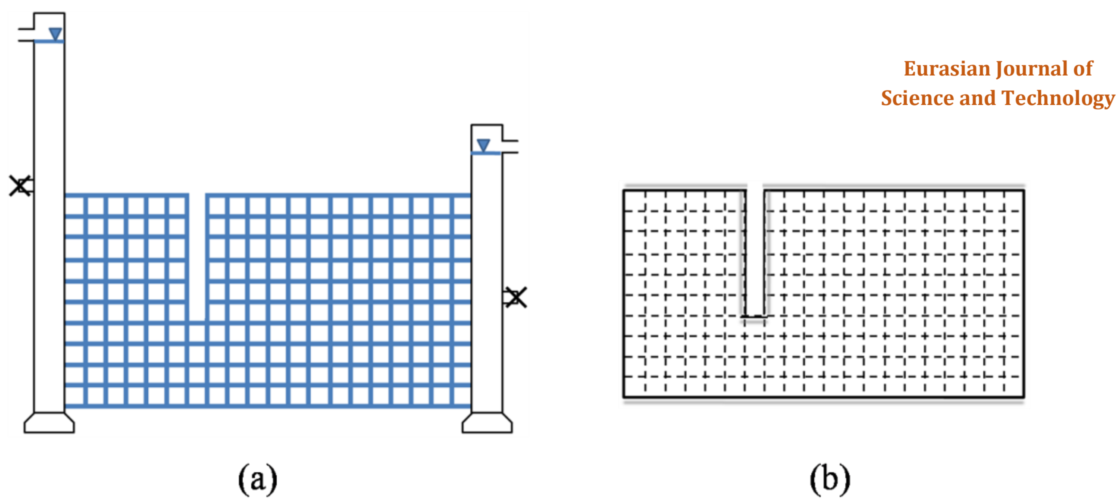

Figure 9 (a) Schematic view of the seal curtain test and (b) computational amplitude of the problem with 10 cm 10 cm divisions

The analytical answer to this problem using the method described in Experiment 1 was compared with the answers obtained using the three methods FD, FE, SPNM, and also the resulting errors are reported in Table 5. Laboratory errors are also listed in Table 5. As can be seen, the error obtained from the SPNM network method and the test results are of acceptable accuracy.

Experiment (4): Flow in free aquifer

This experiment was performed to obtain the water surface profile in open aquifer. Schematic diagrams of the computational range of this aquifer and its test are depicted in Figure 11.

Table 5 Mean percentage of relative error obtained by division using different numerical methods for testing

Mesh size	**TAE**		
	FD	FE	SPNM
2×1 cm	0.28	0.10	0.84

Table 6 Mean percentage of relative error obtained with the division in Experiment (3)

Mesh sizes	TAE of Experiment
2×1 cm	11

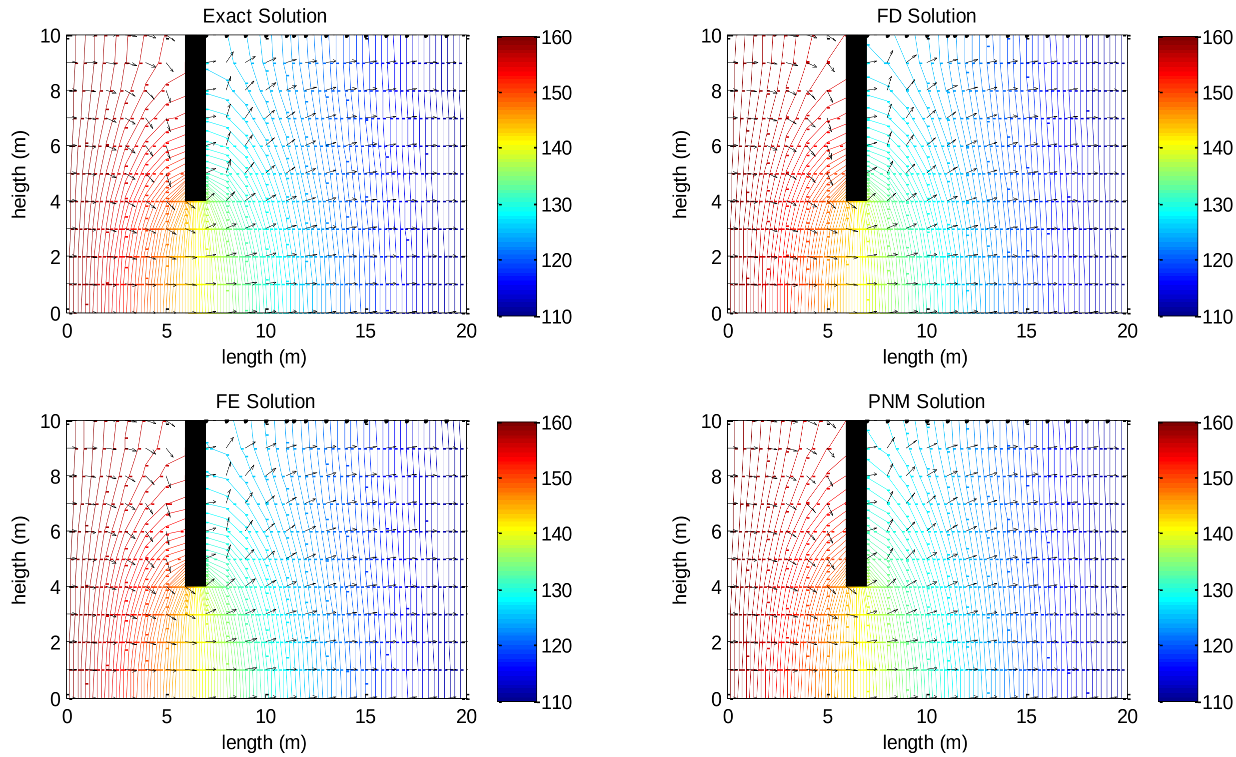

Figure 10 Co-potential lines and velocity vectors obtained through different methods

Figure 11 Water movement model in free aquifer

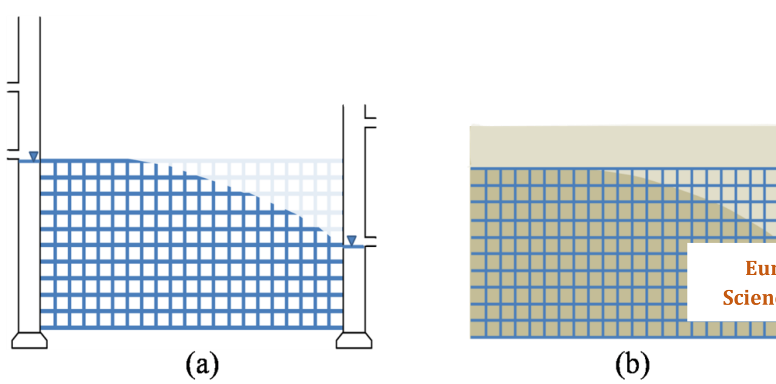

Figure 12 (a) Schematic view of the free aquifer and (b) computational range of the problem with 10 cm divisions

To solve this problem, because the upper boundary, which is the free surface of water, is unknown, the computational range is not clear [20]. There are several methods for solving these problems, two of which were described [21]. The second method is used to solve this problem. For this purpose, the computational range is considered as demonstrated in Figure 13.

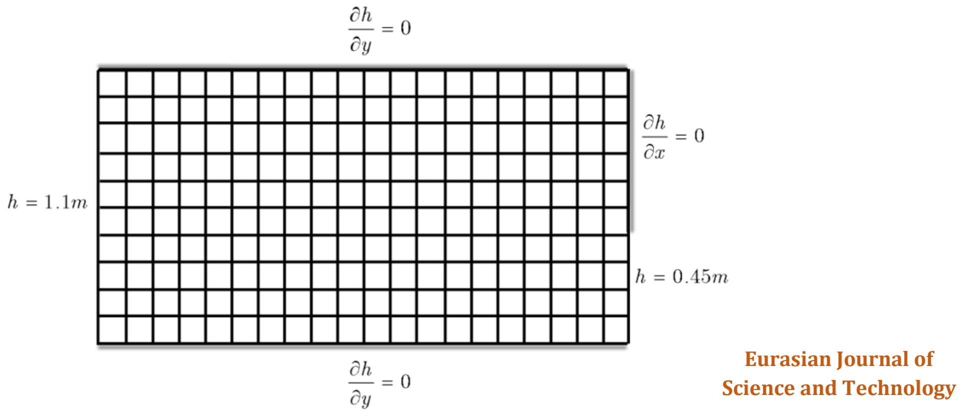

Figure 13 The initial computational range with its boundary conditions for free aquifer solution

After solving this problem, by reducing the head of each node from its height, the amount of pressure of each node is calculated [22]. The nodes that have negative pressure are then removed from the computational domain and the problem is solved twice. This is shown in Figure 14. This method is then repeated until the pressure at any point is negative.

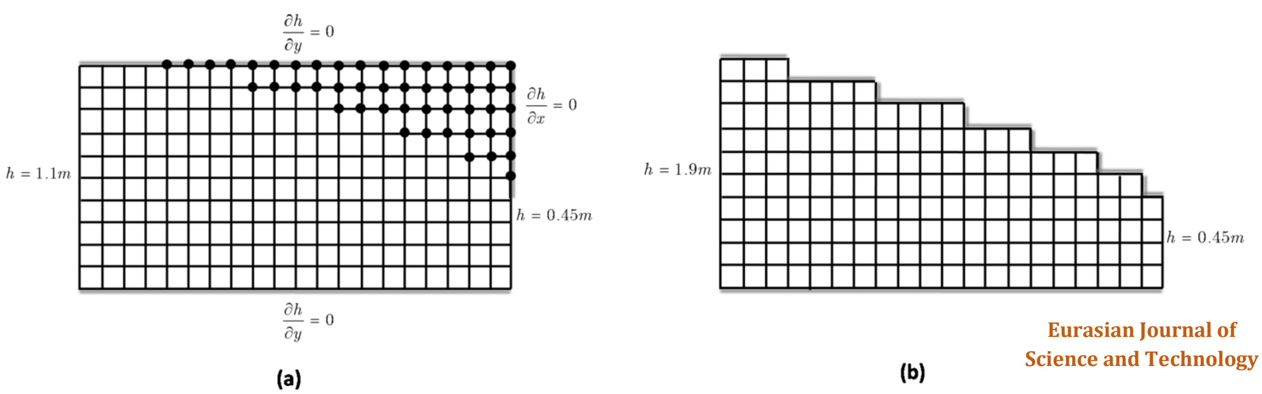

Figure 14 (a) Identify points in the initial domain that have a negative pressure and (b) deleting those points and obtaining a secondary computational range

By comparing the free surface profile of water obtained using FD, FE, and SPNM methods with the analytical answer mentioned, it is possible to understand the accuracy of each method. The profiles obtained by the mentioned methods and the number of errors [23]. As shown in Figure 15, the SPNM numerical method is in good agreement with the FE method and the experimental error is acceptable. In this example, only the FD method does not fit well with the other methods.

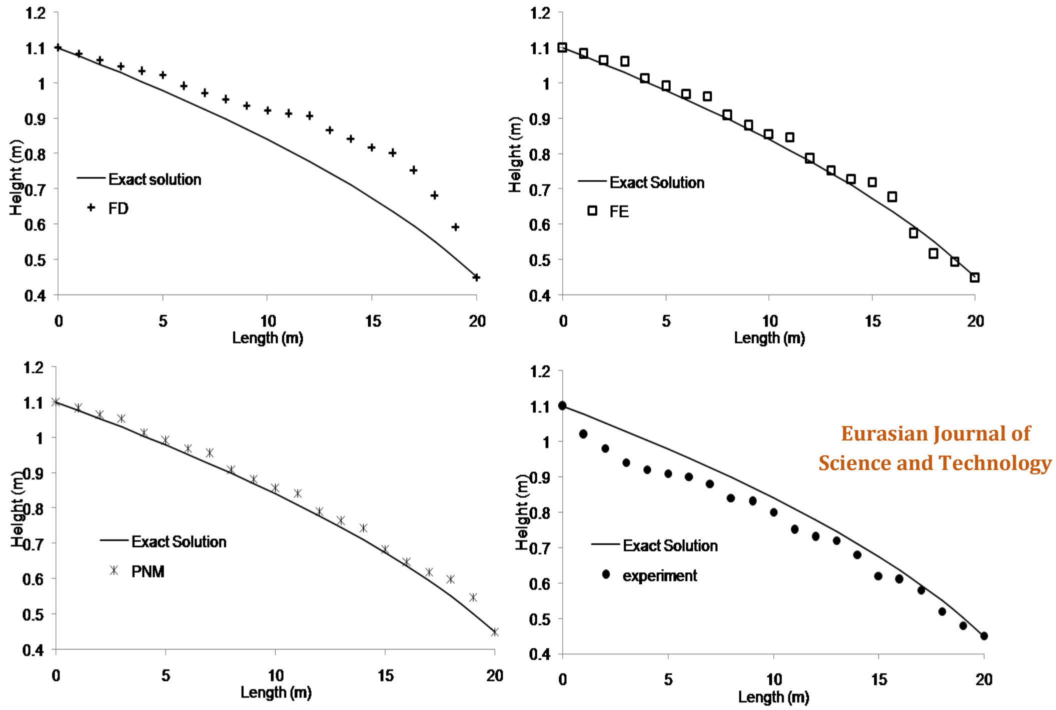

Figure 15 Comparison of free water level using different methods

Table 7 Mean percentage of relative error obtained in calculating free water level

TAE			
FD	FE	SPNM	Experiment
11.7	2.6	2.8	6

Experiment (5): Flow in heterogeneous and heterogeneous aquifer

This experiment was performed to investigate the movement of groundwater in an aquifer that is heterogeneous. This aquifer is shown in Figure 16. The permeability coefficient in the x direction in the middle of the aquifer is half the permeability coefficient of other points. The two upper and lower borders of this aquifer are impenetrable and the left and right borders have a fixed head and are equal to 160 cm and 110 cm, respectively [24].

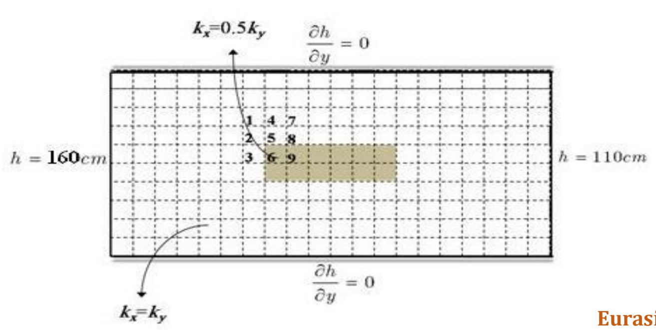

Figure 16 The porous medium intended for Experiment (5)

To construct an experimental model of this problem, the duct diameters of the less permeable areas can be reduced, or the ducts can be lengthened. In this case, due to the fact that the permeability coefficient of the shaded area in the x direction is half the permeability coefficient of other points, the length of the ducts in that area along the x-axis are doubled. The schematic of the experiment is shown in Figure 17.

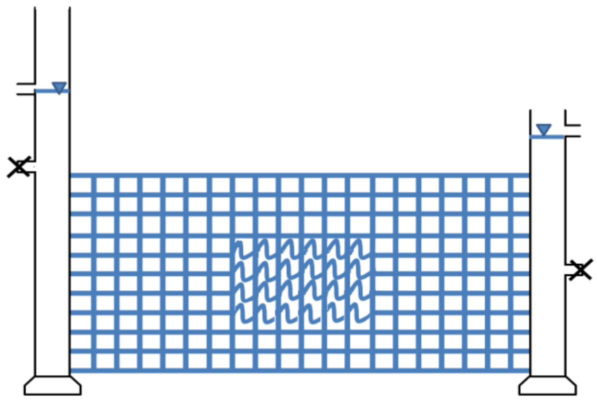

Figure 17 Schematic view of a non-matched porous medium test

As shown in Figure 17, the lengths of the horizontal tubes in the middle section have been increased to simulate a reduction in the hydraulic conductivity. In this Equation (1):

$$3h_5 - 0.5h_8 - h_4 - h_2 - 0.5h_6 = 0 \tag{1}$$

By writing the appropriate equations for all the nodes and solving the obtained device, the answer to the above problem can be achieved. The comparison of the errors of the answers obtained using numerical to laboratory methods is shown in Tables 8 and 9.

Table 8 Mean percentage of relative error calculated using different methods for Experiment (5)

Mesh sizes	TAE		
	FD	FE	SPNM
2×1 cm	0.06	0.19	0.06

Table 9 Mean percentage of relative error obtained in Experiment (5)

Mesh sizes	TAE of Experiment
2×1 cm	9

In Figure 18, the potential lines obtained from different methods are plotted, all of which show the same trend. According to the results, it is clear that the heads read from the experiment are in good agreement with the computational results, which confirms the replacement of this experiment with other experiments such as sand box and so on.

The main advantage of this laboratory method compared to other laboratory methods is that it is easier to implement the change of aquifer properties in the laboratory model. For example, to simulate a change in the hydraulic coefficient of an aquifer, it is only necessary to replace the corresponding tubes in the experimental model with other tubes that simulate the changed characteristics of the aquifer [25].

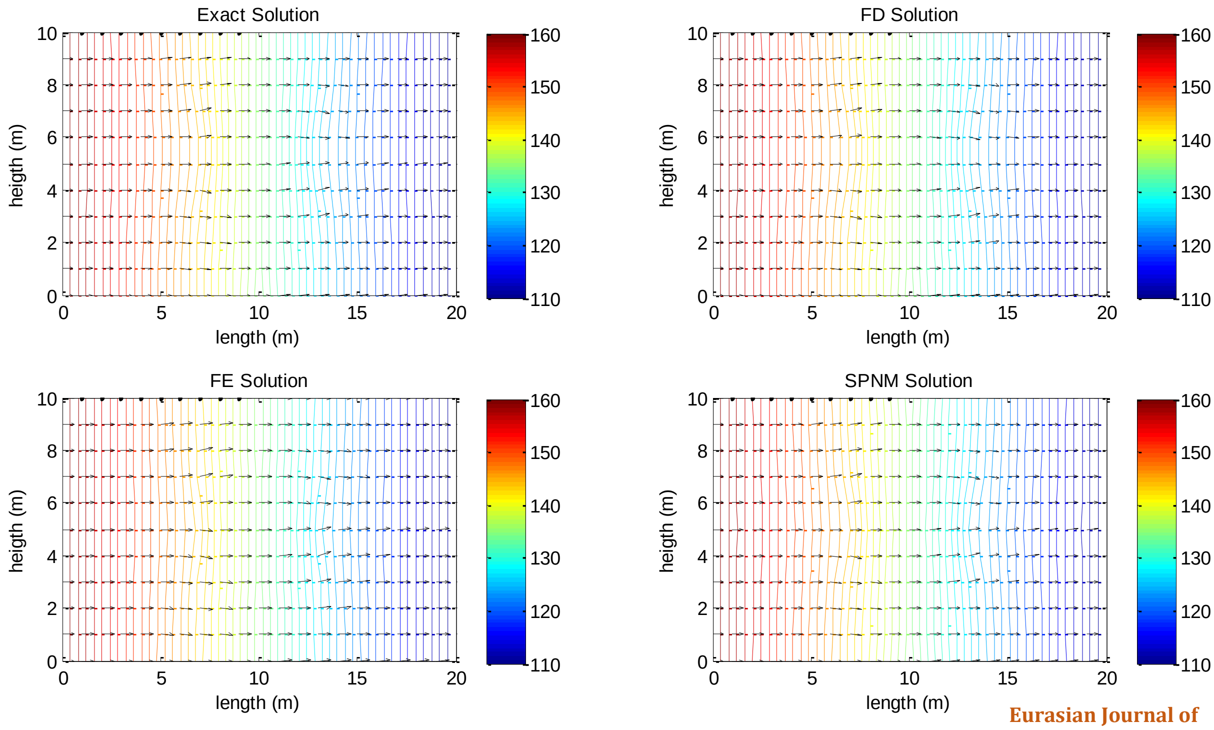

Figure 18 Co-potential lines and velocity vectors obtained through different methods in Experiment (5)

This is while in physical methods, the whole soil of the desired area should be removed and then replaced with suitable soil. In addition to the difficulty of execution, this also causes errors. One of these errors is when to replace the middle parts of the porous medium, the upper parts must initially be removed and then poured back on the replaced middle part.

Due to the re-casting, the properties of the top layer will definitely change. These tests, like all other tests, are not error-free. The following are some sources of error:

- The length of the pipes used in the construction of the network is not the same.
- Inequality of plastic parts that connect plastic crossroads to pipes [26].
- Error in sealing laboratory equipment [27].
- The presence of air bubbles inside the network and its ventilation, which can last up to several hours.
- Impossibility of keeping the head fixed in both input and output power supplies [28].
- Temperature changes during the test and as a result change the viscosity of the fluid.

Conclusion and Recommendations

Due to the linear relationship between the amount of flow inside the pipes at rest and the difference between the two ends of the pipe, the equations obtained to solve the network of pipes will be linear. By writing the flow continuity relation for each node and applying it to all nodes, the equation governing the network of pipes will become a linear system of algebraic equations.

To achieve the objectives of this study, after showing how to construct a coefficient matrix or in other words how to spatially analyze groundwater equations, using the PNM method of the Laplace equation for the case where the analytical answer is available and the answer will be compared with the analytical answer and numerical methods mentioned, and then more complex problems are solved in permanent and non-permanent states, such as the movement of water under the dam despite the impenetrable barrier, and the obtained solution is compared with the answer of numerical methods.

Factors that are effective in solving the groundwater equation using the lattice method and are considered as variables in this study can be summarized as follows:

- The diameter of the pipes;
- Density of pipes in the network;
- How to connect different nodes to each other or in other words network layout After the theoretical foundations of the network method are explained in detail, the PNM model will be developed in the laboratory. For this purpose, checkered grids will be used, which are intersections connected by pipes. This network will be connected to two separate sources upstream and downstream with a specific head and will be measured at specific points of this head network. In order to model the impermeable factors, valves will be installed in this network that by closing these valves, the movement of water inside the pipes can be stopped.

The purpose of this study can be summarized as follows:

- How to construct a coefficient matrix using the PNM method for boundary conditions of the first, second and third types. The effect of some factors such as heterogeneity and heterogeneity of porous medium on the matrix obtained by PNM method.
- Solving groundwater equations using PNM method in different slopes with smooth or curved boundaries, in steady and non-steady state, despite the source of injection and harvesting and comparing the results obtained with finite difference methods and finite elements.

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